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# Classifications of evolution algebras over finite fields

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# Classifications of evolution algebras over finite fields

**Óscar Falcón**

(Joint work with Raúl Falcón and Juan Núñez)

Department of Geometry and Topology. University of Seville.

Seville. January 19, 2016.



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# Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.

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- Evolution algebras.
- Isotopisms of algebras.
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Jianjun Paul Tian, 2004

An  $n$ -dimensional algebra  $E$  over a field  $\mathbb{K}$  is said to be an **evolution algebra** if it admits a basis  $\{e_1, \dots, e_n\}$  such that

- (1)  $e_i e_j = 0$  if  $i \neq j$ ,
- (2)  $e_i e_i = \sum_{j \leq n} a_{ij} e_j$ , for some  $a_{i1}, \dots, a_{in} \in \mathbb{K}$ .

$A = (a_{ij}) \equiv$  Matrix of **structure constants**.



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**Applications:** Non-Mendelian Genetic, Dynamic Systems, Markov Processes, Theory of Knots, Graph Theory and Group Theory.

- Nilpotency and solvability  $\Rightarrow$  Disappearance of population in evolution processes.

References (Tian's Web Page):

<https://www.math.nmsu.edu/~jtian/e-algebra/e-alg-index.htm>



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**Research problem:** Distribution of evolution algebras over finite fields into isomorphism and **isotopism** classes according to the matrices of structure constants.

- **Tool:** Computational Algebraic Geometry.

## Theorem (Casas et al., 2014)

*Every non-zero 2-dimensional complex evolution algebra is isomorphic to exactly one evolution algebra related to one of the next matrix of structure constants*

$$E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 : \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_3 : \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$E_4 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{5_{a,b}} : \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \quad E_{6_c} : \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix},$$

where  $ab \neq 1$ ,  $c \neq 0$  and

- $E_{5_{a,b}} \cong E_{5_{b,a}}$ .
- $E_{6_c} \cong E_{6_{c'}} \Leftrightarrow \frac{c}{c'} = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$ , for some  $k \in \{0, 1, 2\}$ .

# Preliminaries

- Evolution algebras.
- Isotopisms of algebras.
- Algebraic Geometry.



Abraham Adrian Albert

1905-1972

Two algebras  $\mathfrak{a}$  and  $\mathfrak{a}'$  are **isotopic** ( $\simeq$ ) if there exist three regular linear transformations  $f$ ,  $g$  and  $h$  from  $\mathfrak{a}$  to  $\mathfrak{a}'$  such that

$$f(u)g(v) = h(uv), \text{ for all } u, v \in \mathfrak{a}.$$

- The triple  $(f, g, h)$  is an **isotopism** between  $\mathfrak{a}$  and  $\mathfrak{a}'$ .
- To be isotopic is an equivalence relation among algebras.
- $f = g = h \Rightarrow$  **Isomorphism** ( $\cong$ ) of algebras.

**Literature:** **Division algebras** (Albert, Benkart, Bruck, Dieterich, Petersson, Sandler), **Lie algebras** (Falcón, Núñez, Jiménez), **Jordan algebras** (McCrimmon, Oehmke, Petersson, Ple, Thakur), **Alternative algebras** (Babikov, McCrimmon), **Absolute valued algebras** (Albert, Cuenca), **Structural algebras** (Allison).

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# Algebraic Geometry

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- A **Gröbner basis** of an ideal  $I$  is any subset  $G$  of polynomials in  $I$  whose leading monomials generate the initial ideal.
- It is **reduced** if all its polynomials are monic and no monomial of a polynomial in  $G$  is generated by the leading monomials.

# Algebraic Geometry

Let  $I$  be an ideal in  $\mathbb{F}_p[\underline{x}]$ .

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$$\{f^m \in I \Rightarrow f \in I\}, \text{ for all } f \in \mathbb{F}_p[\underline{x}] \text{ and } m \in \mathbb{N}.$$

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## Theorem

*If  $I$  is zero-dimensional and radical, then*

$$|\mathcal{V}(I)| = \dim_{\mathbb{F}_p} \mathbb{F}_p[\underline{x}]/I$$

*and coincides with the number of standard monomials of  $I$ .*

Reduced Gröbner bases play a fundamental role in the computation of  $|\mathcal{V}(I)|$ .

## Theorem (Lakshman and Lazard, 1991)

*The complexity of computing the reduced Gröbner basis of a zero-dimensional ideal is  $d^{O(n)}$ , where*

- *$d$  is the maximal degree of the polynomials of the ideal.*
- *$n$  is the number of variables.*

# Isotopisms of evolution algebras.

## Lemma

Let  $E$  and  $E'$  be two isotopic evolution algebras of respective matrices of structure constants  $A = (a_{ij})$  and  $A' = (a'_{ij})$ . Let  $(f, g, h)$  be an isotopism between both algebras related, respectively, to the matrices  $F = (f_{ij})$ ,  $G = (g_{ij})$  and  $H = (h_{ij})$ . Then,

a)  $\sum_{j \leq n} f_{ij} g_{ij} a'_{jk} = \sum_{j \leq n} a_{ij} h_{jk}$  for all  $i, k \leq n$ .

b)  $\sum_{k \leq n} f_{ik} g_{jk} a'_{kl} = 0$ , for all  $i, j, l \leq n$ .

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b)  $\sum_{k \leq n} f_{ik} g_{jk} a'_{kl} = 0$ , for all  $i, j, l \leq n$ .

### Proof.

- a) Let  $\{e_1, \dots, e_n\}$  and  $\{e'_1, \dots, e'_n\}$  be respective bases of  $E$  and  $E'$ . Let  $i \leq n$ . Then,

$$\begin{aligned} \sum_{j, k \leq n} f_{ij} g_{ij} a'_{jk} e'_k &= \sum_{j \leq n} f_{ij} e'_j \cdot \sum_{j \leq n} g_{ij} e'_j = f(e_i)g(e_i) = \\ &= h(e_i e_i) = h\left(\sum_{j \leq n} a_{ij} e_j\right) = \sum_{j, k \leq n} a_{ij} h_{jk} e'_k \end{aligned}$$

□

## Lemma

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b)  $\sum_{k \leq n} f_{ik} g_{jk} a'_{kl} = 0$ , for all  $i, j, l \leq n$ .

### Proof.

b) Let  $\{e_1, \dots, e_n\}$  and  $\{e'_1, \dots, e'_n\}$  be respective bases of  $E$  and  $E'$ . Let  $i, j \leq n$  be such that  $i \neq j$ . Then,

$$\begin{aligned} \sum_{k, l \leq n} f_{ik} g_{jk} a'_{kl} e'_l &= \sum_{k \leq n} f_{ik} e'_k \cdot \sum_{k \leq n} g_{jk} e'_k = f(e_i)g(e_j) = \\ &= h(e_i e_j) = 0 \end{aligned}$$

□

Let  $E$  be an evolution algebra. The **annihilator** of  $E$  is

$$\text{Ann}(E) = \{x \in E \mid xE = 0\}.$$

### Lemma

*Let  $E$  and  $E'$  be two isotopic evolution algebras and let  $(f, g, h)$  be an isotopism between them. Then,*

$$f(\text{Ann}(E)) = g(\text{Ann}(E)) = \text{Ann}(E').$$

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$$f(\text{Ann}(E)) = g(\text{Ann}(E)) = \text{Ann}(E').$$

### Proof.

Let  $x \in \text{Ann}(E)$ . Then,

$$f(x)E' = f(x)g(E) = h(xE) = h(0) = 0.$$

Thus,  $f(\text{Ann}(E)) \subseteq \text{Ann}(E')$ . The reciprocal holds similarly. The identity with  $g$  also holds analogously.  $\square$

## 2-dimensional complex evolution algebras

$$\begin{aligned} E_1 &: \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & E_2 &: \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & E_3 &: \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ E_4 &: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & E_{5_{a,b}} &: \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, & E_{6_c} &: \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}. \\ & & ab \neq 1 & & c \neq 0 \end{aligned}$$

### Proposition

*There are three isotopism classes in the set of 2-dimensional complex evolution algebras:*

- a)  $E_1 \simeq E_4$ .
- b)  $E_2 \simeq E_3$ .
- c)  $E_{5_{a,b}} \simeq E_{6_c}$ , for all  $a, b, c$ .

## 2-dimensional complex evolution algebras

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### Proof.

$$c) F = G = \begin{pmatrix} \sqrt{c-a} & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ \frac{1-ab}{b} & 1 \end{pmatrix}.$$

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## Proof.

d)  $E_2 \not\simeq E_1 \not\simeq E_5$ .

$$\text{Ann}(E_1) = \langle e_2 \rangle, \quad \text{Ann}(E_2) = \text{Ann}(E_5) = \emptyset.$$

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**Proof.** d)  $E_2 \not\simeq E_{6_c}$ .

$$\begin{cases} f_{12}g_{22} = f_{22}g_{12} = f_{11}g_{21} = f_{21}g_{11} = 0, \\ h_{11} = f_{12}g_{12} = f_{22}g_{22}, \\ h_{12} = f_{11}g_{11} = f_{21}g_{21} \end{cases} \Rightarrow h_{11} = h_{12} = 0 \Rightarrow |H| = 0!!!$$

# Procedures

- Field:  $\mathbb{K}$ .
- Sets of  $n^2$  variables:

$$\mathfrak{A}_n = \{a_{ij} : i, j \leq n\},$$

$$\mathfrak{F}_n = \{f_{ij} : i, j \leq n\},$$

$$\mathfrak{G}_n = \{g_{ij} : i, j \leq n\},$$

$$\mathfrak{H}_n = \{h_{ij} : i, j \leq n\}.$$

- Multivariate polynomial rings:

$$\mathbb{K}[\mathfrak{A}_n \cup \mathfrak{F}_n] \quad \text{and} \quad \mathbb{K}[\mathfrak{A}_n \cup \mathfrak{F}_n \cup \mathfrak{G}_n \cup \mathfrak{H}_n].$$

- Matrices:

$$F = (f_{ij}), \quad G = (g_{ij}), \quad H = (h_{ij}).$$

- Let  $\mathfrak{E}_n^{\mathbb{K}}$  be the  $n$ -dimensional algebra with basis  $\beta_n = \{e_1, \dots, e_n\}$  such that

$$e_i e_j = \sum_{k=1}^n a_{ijk} e_k, \quad \text{for all } i, j \leq n.$$

# Isomorphisms between two $n$ -dimensional evolution algebras over $\mathbb{F}_p$ .

```
1: procedure ISOM( $n, p, A, A'$ )
2:   for  $i \leftarrow 1, n$  do
3:     for  $k \leftarrow 1, n$  do
4:        $l = l + (f_{ik}^p - f_{ik}) + (\alpha_{ik}^p - \alpha_{ik}) + (\alpha'_{ik} - \alpha_{ik})$ ;
5:        $\text{pol}_1 = 0$ 
6:       for  $j \leftarrow 1, n$  do
7:          $\text{pol}_1 = \text{pol}_1 + (f_{ij}^2 \alpha'_{jk} - \alpha_{ij} f_{jk})$ ;
8:          $\text{pol}_2 = 0$ 
9:         for  $l \leftarrow 1, n$  do
10:           $\text{pol}_2 = \text{pol}_2 + f_{il} f_{jl} \alpha'_{lk}$ 
11:        end for
12:         $l = l + \text{pol}_2$ ;
13:      end for
14:       $l = l + \text{pol}_1$ ;
15:    end for
16:  end for
17:  for  $i \leftarrow 1, \text{size}(A)$  do
18:     $l = l + (\alpha_{A[i][1]A[i][2]} - A[i][3])$ ;
19:  end for
20:  for  $i \leftarrow 1, \text{size}(A')$  do
21:     $l = l + (\alpha'_{A'[i][1]A'[i][2]} - A'[i][3])$ ;
22:  end for
23:   $l = l + (\det(F)^{p-1} - 1)$ ;
24:   $l = \text{Gröbner}(l)$ ;
25:  return  $|\mathcal{V}(l)|$ ;
26: end procedure
```

# Isotopisms between two $n$ -dimensional evolution algebras over $\mathbb{F}_p$ .

```
1: procedure ISOT( $n, p, A, A'$ )
2:   for  $i \leftarrow 1, n$  do
3:     for  $k \leftarrow 1, n$  do
4:        $I = I + (f_{ik}^p - f_{ik}) + (g_{ik}^p - g_{ik}) + (h_{ik}^p - h_{ik}) + (a_{ik}^p - a_{ik}) + (a'_{ik}^p - a'_{ik})$ ;
5:        $pol_1 = 0$ 
6:       for  $j \leftarrow 1, n$  do
7:          $pol_1 = pol_1 + (f_{ij}g_{ij}a'_{jk} - a_{ij}h_{jk})$ ;
8:          $pol_2 = 0$ 
9:         for  $l \leftarrow 1, n$  do
10:           $pol_2 = pol_2 + f_{il}g_{jl}a'_{lk}$ 
11:        end for
12:         $I = I + pol_2$ ;
13:      end for
14:       $I = I + pol_1$ ;
15:    end for
16:  end for
17:  for  $i \leftarrow 1, \text{size}(A)$  do
18:     $I = I + (a_{A[i][1]A[i][2]} - A[i][3])$ ;
19:  end for
20:  for  $i \leftarrow 1, \text{size}(A')$  do
21:     $I = I + (a'_{A'[i][1]A'[i][2]} - A'[i][3])$ ;
22:  end for
23:   $I = I + (\det(F)^{p-1} - 1) + (\det(G)^{p-1} - 1) + (\det(H)^{p-1} - 1)$ ;
24:   $I = \text{Gröbner}(I)$ ;
25:  return  $|\mathcal{V}(I)|$ ;
26: end procedure
```

# Finite dimensional evolution algebras.

## 2-dimensional evolution algebras.

### Lemma

Let  $A$  and  $A'$  be the matrix of structure constants of two isotopic 2-dimensional evolution algebras and let  $(f, g, h)$  be an isotopism between them. Let  $F = (f_{ij})$  and  $G = (g_{ij})$  be the matrices related, respectively, to  $f$  and  $g$ . If  $|A'| \neq 0$ , then exactly one of the next two assertions holds.

a)  $f_{11} = f_{22} = g_{11} = g_{22} = 0$ .

b)  $f_{12} = f_{21} = g_{12} = g_{21} = 0$ .

**Proof.** The reduced Gröbner basis of the ideal related to our isotopism contains the next two generators

$$\begin{cases} f_{22} \cdot g_{12} \cdot |A'| = 0, \\ f_{12} \cdot g_{22} \cdot |A'| = 0. \end{cases}$$

Hence,  $f_{22} = g_{22} = 0$  or  $f_{12} = g_{12} = 0$ .

## 2-dimensional evolution algebras.

### Lemma

Let  $A$  and  $A'$  be the matrix of structure constants of two isotopic 2-dimensional evolution algebras and let  $(f, g, h)$  be an isotopism between them. Let  $F = (f_{ij})$  and  $G = (g_{ij})$  be the matrices related, respectively, to  $f$  and  $g$ . If  $|A'| \neq 0$ , then exactly one of the next two assertions holds.

- a)  $f_{11} = f_{22} = g_{11} = g_{22} = 0$ .
- b)  $f_{12} = f_{21} = g_{12} = g_{21} = 0$ .

**Proof.** In both cases,

$$\begin{cases} a'_{11}f_{11}g_{21} = 0, \\ a'_{12}f_{11}g_{21} = 0, \\ a'_{11}f_{21}g_{11} = 0, \\ a'_{12}f_{21}g_{11} = 0. \end{cases} \Rightarrow \begin{cases} f_{22} = g_{22} = 0 \Rightarrow f_{11} = g_{11} = 0, \\ f_{12} = g_{12} = 0 \Rightarrow f_{21} = g_{21} = 0. \end{cases}$$

## 2-dimensional evolution algebras.

### Theorem

*There are seven non-zero 2-dimensional evolution algebras over  $\mathbb{F}_2$  up to isomorphisms:*

$$E_1^2 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2^2 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3^2 : \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_4^2 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$E_5^2 : \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_6^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_7^2 : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

### Theorem

*There are three non-zero 2-dimensional evolution algebras over  $\mathbb{F}_2$  up to isotopisms:*

- a)  $E_1^2 \simeq E_2^2$ .
- b)  $E_3^2 \simeq E_7^2$ .
- c)  $E_4^2 \simeq E_5^2 \simeq E_6^2$ .

## 2-dimensional evolution algebras.

### Theorem

Let  $p \in \{3, 5\}$ . There are six non-zero 2-dimensional evolution algebras over  $\mathbb{F}_p$  up to isomorphisms:

$$E_1^p : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2^p : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3^p : \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_4^p : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$E_5^p : \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_6^p : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

### Theorem

Let  $p \in \{3, 5\}$ . There are three non-zero 2-dimensional evolution algebras over  $\mathbb{F}_p$  up to isotopisms:

- $E_1^p \simeq E_2^p$ .
- $E_3^p$ .
- $E_4^p \simeq E_5^p \simeq E_6^p$ .

## 2-dimensional evolution algebras.

### Theorem

Let  $A = (a_{ij})$  be the matrix of structure constants of a 2-dimensional evolution algebra  $E$  over a field  $\mathbb{K}$ . If  $|A| \neq 0$ , then,  $E$  is isotopic to the 2-dimensional evolution algebra over  $\mathbb{K}$  of related matrix of structure constants

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Proof.** It is enough to consider the isotopism  $(f, g, h)$  of matrices

$$F = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}.$$

□

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# Many thanks!!

## Classifications of evolution algebras over finite fields

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